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An SPH Model to Simulate the Dynamic Behavior of Shear Thickening Fluids

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Abstract

While significant research has been dedicated to the simulation of fluids, not much attention has been given to exploring new interesting behavior that can be generated with different types of non-Newtonian fluids with non-constant viscosity. Going in this direction, this paper introduces a computational model for simulating the most interesting phenomena observed in non-Newtonian shear thickening fluids, which are fluids where the viscosity increases with increased stress. These fluids have unique and unconventional behavior, and they often appear in real world scenarios such as when sinking in quicksand or when experimenting with popular cornstarch and water mixtures. The fluid exhibits unique phase changes between solid and liquid states, great impact resistance in its solid state and strong hysteresis effects. Our proposed approach builds on existing non-Newtonian SPH fluid models in computer graphics and introduces an efficient history-based stiffness term that is essential to produce the most interesting types of shear thickening phenomena. The history-based stiffness is formulated through the use of fractional derivatives, leveraging the fractional calculus ability to depict both the viscoelastic behavior and the history effects of history-dependent systems. Simulations produced by our method are compared against real experiments and the results demonstrate that the proposed model successfully captures key phenomena specific to discontinuous shear thickening fluids.

Keywords: fluid simulation, shear thickening fluids, fractional derivatives

1 Introduction

In the field of rheology, the mechanics of fluids is typically characterized by a viscosity function, where the viscosity is defined as the ratio of shear stress to shear rate in a steady state flow. For a

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Newtonian fluid like water, this viscosity is a single-valued constant, but in non-Newtonian fluids this viscosity may be a function of shear rate and other parameters. Dense suspensions of hard particles are known to be shear thickening, meaning that the viscosity function increases with shear rate over some range.

Shear thickening fluids (STFs) form a sub-category of non-Newtonian fluids exhibiting unconventional behavior that is not seen in other Newtonian or non-Newtonian fluids. These fluids harden when agitated by strong forces and soften in the absence of any forces. Moreover, they are history-dependent fluids in the sense that they keep their solid-like state for a given amount of time even after the forces are applied, i.e., their memory only fades away after some time. See Figure 1 for an example obtained with our simulation model.

The most known example of shear thickening fluids is probably the cornstarch and water mixture. The odd behavior of this fluid often attracts public attention and videos showing experiments involving mixtures of cornstarch and water are widely spread over the Internet. In these experiments, it is possible to witness the dramatic impact resistance of the fluid to the point that a person can run on top of its surface without sinking [1].

Another STF that has received some attention in several movies is quicksand. Innovative applications of STFs have also been developed. For example, the United States Army Research Laboratory has developed a "Liquid Body Armor", an armor suit that has layers of shear thickening fluid and Kevlar mixture. The fabric made of STF is highly resistant to penetration when impacted by a spike, knife or bullet, without compromising its weight, comfort or flexibility in normal conditions [2] [3].

Despite the growing interest in the properties of shear thickening fluids, achieving their most peculiar behavior in simulation has not yet received specific attention. For instance, while several models exist to simulate viscoelastic fluids, viscoelastic and shear thickening are non-overlapping subcategories of non-Newtonian behavior. Our proposed model addresses a subcategory more specifically referred to as *discontinuous shear thickening fluids* in the literature, which includes the most interesting phenomena observed in cornstarch and water mixtures (shown in Figures 1 and 5.)

Our contribution in this paper is significant to computer graphics in several forms: it demonstrates new simulation effects that can be achieved with models based on fractional derivatives, it shows that key STF phenomena can be addressed with reasonably simple SPH-based techniques, and it presents visual simulations of some of the most peculiar behavior of an interesting class of materials.

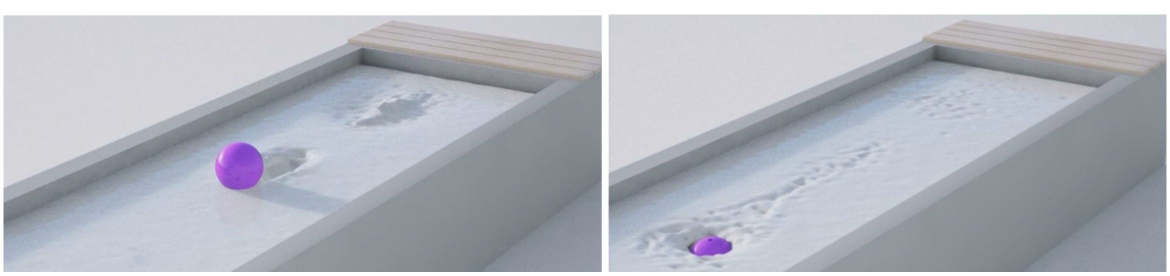

Figure 1: This scenario illustrates the typical behavior of a shear thickening fluid: while the bowling ball hits the fluid with high energy the bowling ball bounces (left image), and only when the energy starts to dissipate the bowling ball will then gradually sink, as in a standard fluid (right image).

2 Related Work

The simulation of shear thickening fluids involves concepts from different disciplines. We first review previous work in physics that discuss the behavior of shear thickening fluids. We then review a group of rheology and physics papers that use fractional calculus to describe viscoelasticity and the behavior of non-Newtonian fluids. Finally, we review previous work in computer graphics that are related to simulation of non-Newtonian or viscoelastic fluids.

2.1 Physics and Rhelology

While models exist for shear thickening, including the hydrocluster model [4] [5] and the dilatancy model [6], no physics-based simulations have yet been able to show any of the peculiar phenomena that shear thickening fluids are known for. The problem seems to be that traditional rheological models like those mentioned above are defined in a steady-state flow, and it is assumed that this is a complete rheology that can describe different flows. However, more recent work has shown that models for dense suspensions require more features to fully characterize the different phenomena they exhibit.

One of these model features required to explain phenomena is hysteresis (history effects). Using a modified Herschel-Bulkley model, Deegan [7] showed that indentations in the surface of a vibrated shear thickening fluid can only be stable or grow if there is hysteresis in the viscosity. Any single-valued viscosity function can produce a drag force that slows the flattening of the indentations, but regardless of the functional form, cannot cause the indentations to grow or remain stable due to vibration. Instead, if the drag force has a hysteresis with a large enough difference on the up- and down-cycles of the oscillation to balance or overcome the forces of gravity and surface tension, then there can be a net force to allow the indentations to stabilize or grow. Kann et al. [8] showed that oscillations in the velocity of an object sinking in shear thickening fluid can similarly be explained by hysteresis in the viscosity. A sinking object is driven by gravity, and slowed by a drag force related to the viscosity. Any single-valued function where the force increases with viscosity results in a velocity monotonically approaching a steady state where the drag force balances gravity. Oscillations in the velocity can be obtained if there is hysteresis in the viscosity such that there is a higher drag force for a history of decreasing velocity.

More recent experiments motivated by the simulations shown in this paper have shown that the hysteresis can be broken down into two separate features: the time it takes for the suspension to solidify in response to impact, and the time it takes to melt back into a fluid when the stress is removed. The melting can be described by a relaxation time which is not determined by the viscosity of the suspensions (as is the traditional case of a generalized Newtonian fluid), which allows the high viscosity suspensions to relax quickly (on the order of 1 second) [9]. The solidification has been shown to result from colliding particles collecting to form a transient solid, with a front that develops at the impact site and propagates ahead of the impacter [10]. The material can support a load like a solid once this front propagates to another boundary [11]. These experiments have also shown that the impact response is more accurately characterized by a velocity-dependent stiffness (rather than viscosity), and that these properties can accurately be used to model the ability of people to run on the surface of cornstarch and water [12].

2.2 Fractional Calculus and Viscoelasticity

Even though our model is the first to use fractional calculus for the simulation of shear thickening fluids in the realm of computer graphics, the approach comes from existing mathematical models developed in the physics and rheology fields. Fractional calculus is popular for describing the complex dynamics of relaxation, oscillation and viscoelastic behavior seen in non-Newtonian fluids. In general, fractional calculus is employed by replacing the time-derivatives of strain and stress in well-established ordinary models, by derivatives of fractional order. This results in a more appropriate model for fluids that exhibit both viscous and elastic behavior.

A detailed literature review on the attempts of using fractional derivative models to non-Newtonian fluids in order to obtain analytic solutions is presented in [13]. The first work to adopt fractional derivatives to the problem of viscoelasticity was presented by Germant [14]. Following its introduction, Bagley and Torvik [15] demonstrated predictive constitutive relations based on fractional derivatives for the viscoelasticity of coiling polymers. Makris and Constantinou [16] were able to successfully fit a fractional derivative based on the Maxwell model to the experimental data, hence showing that a fractional Maxwell model can replace the Maxwell model for silicon fluids. Heibiga [17] also used a one-dimensional fractional derivative based on the Maxwell model for the linear viscoelastic response of some polymers in the glass transition. Palade [18] reduced the constitutive equation for the incompressible fluid to the linear fractional derivative Maxwell model in the context of small deformations.

2.3 Computer Graphics

Fractional derivatives have been introduced in computer graphics with a deformation model for underwater cloth simulation. The approach achieves a history-based surface model that does not require the simulation of the body of water surrounding the cloth model [19]. In this paper, we employ the history effects of fractional derivatives for the purpose of simulating the interesting behavior of discontinuous shear thickening fluids.

Shear thickening behavior has been addressed in previous work with a modified Herschel-Bulkley model [20]. While comprehensive results have been demonstrated, they have not addressed the unusual peculiar phenomena associated with shear thickening fluids. The modified Herschel-Bulkley model is an abstract general form for a bulk rheology that can include shear thinning and shear thickening in a mathematical sense, but this model misses almost all of the important features of real-world shear thickening fluids, including the critical stresses that set the range of shear thickening in a viscosity function, so that shear thickening appears as a transition from a low-viscosity state to a high-viscosity state at a critical shear rate [6], as well as the hysteresis and velocity-dependent stiffness mentioned in Section 1.1. In particular, the modified Herschel-Bulkley model has been used as a specific example when proving that a viscosity function without hysteresis cannot produce the stable or growing indentations achieved in our results [7]. The key physics that has to be taken into account to describe the phenomena shown in Figures 1 and 5 is hysteresis in the constitutive relation [7, 21], and the velocity-dependent stiffness [11], and this is the focus of our proposed model which is able to produce visually successful simulations of the most interesting (and peculiar) phenomena observed in shear thickening fluids. More recently, Non-Newtonian fluids linking stiffness to the shear rate [22] have been proposed achieving shear-thickening and shear-thinning behavior, but again without considering hysteresis and not addressing the specific phenomena that is the target of this paper.

Other relevant papers to our work are the ones addressing melting and flow of materials. Terzopoulos and Fleischer [23] developed a method that can simulate non-rigid objects that are capable of heat conduction, thermoelasticity and melting. The particles that constitute the material are connected by elastic springs and, in order to model the melting effect, the stiffness of the springs decreases as the temperature rises, and the particles break free when the springs disappear. Desbrun and Gascuel [24] also focused on the flow of soft substances. They employed a hybrid model by combining an implicit surface model with a particle system. Successful results were presented on the animation of soft materials that can undergo fusion.

Carlson and Turk [25] took on the challenge of animating materials with both fluid and solid-like behavior. The target was on materials that can melt, flow and solidify. A modification of the marker-and-cell method was used to deal with the computational barrier of simulating highly viscous fluids, since the solidity of the materials was in fact achieved by simulating a highly viscous fluid. No elastic elements were added to the fluid equations. The main difference with our proposed model is that we achieve a material which transitions from fluid-like behavior to solid-like behavior according to the energy in the system, as demonstrated with the simulated fluid supporting a heavy object during its high-energy phase (Figure 1) and with the induced behavior of building and supporting structures on itself (Figure 5).

Goktekin et al. [26] presented a method that can animate viscoelastic fluids such as mucus, liquid soap, toothpaste and clay. The method introduced elements of elasticity into an Eulerian implementation of the Navier-Stokes equations. The total strain is defined as the sum of elastic and plastic strains. The work shows impressive results demonstrating the capabilities of partially maintaining the original object shape during interactions, an expected property of viscoelastic fluids. In comparison, shear thickening fluids behave much like solids under high energy forces, and they automatically transition from a rigid to fluid behavior as energy dissipates. This transition is the key difference modeled in our method. This difference between a typical viscoelastic behavior and our model is depicted in Figure 10, where the shear thickening behavior makes the fluid to mostly preserve its shape while clearly bouncing up a surface in reaction to falling on it, something that is not observed from regular models for viscoelastic behavior.

Clavet et al. [27] developed a powerful and simple method that can efficiently and realistically simulate viscoelastic fluids. The method combines a simplified Smoothed Particle Hydrodynamics (SPH) model with a mass-spring system. The SPH is responsible for the fluid behavior while the springs are responsible for the elasticity and plasticity effects that make the material more solid-like. The method computes an additional pressure term named near-pressure, along with the conventional pressure term calculated through SPH. This term is used to solve the clustering problem which is a common problem in SPH implementations. Furthermore, the nearby particles are also connected to each other by springs. These springs have varying rest lengths and are removed when the distance between a pair of particles is greater than a threshold. The use of adaptive springs along with the double density relaxation method results in a very strong animation tool for viscoelastic models. This method constitutes the starting point for our proposed shear thickening fluid animation model.

Desbrun and Cani [28] demonstrated the first application of SPH in Computer Graphics for the animation of inelastic deformable bodies with a wide range of stiffness and viscosity. Over the years SPH has been established as one major approach for fluid animation in computer graphics, being applied to different applications such as for simulating foam or bubbles [29]. Several extensions have been proposed, for example, exploring vorticity diffusion with vorticity preservation for viscous fluids [30], achieving efficient divergence-free velocity fields [31], and addressing viscous SPH fluids that can generate coiling and buckling phenomena [32].

In addition to SPH models, position-based methods have also become popular in the computer graphics community because of the improved speed, stability and control offered by the approach, even if the results may not be as accurate as force-based methods. Position-based methods have led to the concept of Position-Based Dynamics (PBD) [33], and formulations specific to fluid simulation have been proposed leading to Position-Based Fluids [34]. Impressive results have also been demonstrated for fluids with high viscoelasticity conformation constraints which can be solved efficiently within PBD frameworks [35]. While our approach to achieve the proposed shear thickening effects is based on the specific model of Clavet et al. [27], we believe our introduced concepts can be integrated in position-based models. We discuss this possibility of future work in Section 4.5.

In summary, the behavior of shear thickening fluids has not yet been fully explored and their simulation is an interesting animation problem. At the same time, the ability of fractional calculus to describe history-based behavior is well established in rheology. Motivated by these facts, this paper builds on a non-Newtonian fluid model from computer graphics [27] and introduces a fractional calculus model to efficiently simulate the most interesting behavior of discontinuous shear thickening fluids.

3 Proposed Model

The starting point of our shear thickening simulation model is based on the model presented by Clavet et al. [27]. We give below a summary of this model, and we then focus on describing our solution for achieving the shear thickening behavior.

3.1 Preliminaries

Clavet et al. [27] simulate viscoelastic fluids using a Lagrangian model that combines SPH forces with a dynamic mass-spring system. The density of a particle is computed based on the distance to other particles in its vicinity. The pair-wise distances are smoothed through a smoothing kernel function. The densities are then used to calculate the pressures at particle locations. Two density terms namely near-density and far-density are calculated in slightly different ways and this procedure is called double density relaxation. The additional near-density term is employed to counteract the clustering problem and to create better surface tension. The viscosity force between two particles is calculated based on the difference between their velocities, in accordance with the Navier-Stokes equations. The difference in velocities has both linear and quadratic effects. The elasticity component originates from the spring-mass system.

Particle pairs that are close to each other up to a given distance are connected to each other by elastic springs, and the spring force is given by:

$$\boldsymbol{F}_s = -k_{min}\,\boldsymbol{x}, \tag{1}$$

where k_{min} is the spring constant. The springs only exist between pairs of particles that are closer to each other than a given distance. If two particles move farther away from each other, then the spring connecting them is removed. The rest lengths of the springs are also updated at every simulation step. The addition and removal of springs along with the rest length updates create the effect of plasticity. The displacements originating from the SPH and spring forces are applied through a prediction-relaxation approach. For more details about this model we refer the reader to the work of Clavet et al. [27].

3.2 Shear Thickening Behavior

The existence of viscosity, elasticity and plasticity terms lead to the simulation of viscoelastic materials or non-Newtonian fluids. However, these components are not sufficient for creating a shear thickening fluid.

Shear thickening fluids have two very important additional characteristics when compared to viscoelastic fluids. First, they change their state from liquid-like to solid-like when subjected to great forces. As long as forces are present, the solid-like state is preserved. In the absence of forces, the material will soften and return to its liquid-like structure. Second, shear thickening fluids are history-dependent materials. The present state of these materials is defined in terms of past states, and the materials remember the past forces that affected them up to some extent. When a shear thickening fluid is in its solid-like state and all forces are ceased, it will take some time to go back to its liquid-like state. Likewise, when the fluid is in its liquid state and is agitated by forces, it will slowly move from liquid to solid state over time.

In our proposed approach, the most critical factor that will transform a viscoelastic fluid into a shear thickening fluid is the inclusion of history-based spring elements. These elements have a history-based stiffness constant that injects the information of the past into the system. The history-based stiffness terms are achieved by multiplying the stiffness constant of the dynamic springs by the magnitude of the fractional derivative of the position of the connected particle. The history-based spring element is given by:

$$\boldsymbol{F}_{hist} = -k_{hist}\,\|D^q\boldsymbol{x}\|\,\boldsymbol{x}, \tag{2}$$

where $k_{hist}\,\|D^q\boldsymbol{x}\|$ is the history-based stiffness, $D^q\boldsymbol{x}$ is the q^{th} order time derivative of the position and q is a non-integer between 0 and 1.

The approach used to achieve the history effects relies on the properties of fractional calculus. As discussed in Section 2.2, fractional calculus is widely used for the description of viscoelastic phenomena and it provides an appropriate tool for the simulation of history effects.

The reason for achieving history effects can be clearly seen by the fractional derivative formulation, which in its computational form involves a summation of past integer derivative terms. The numerical formulation adopted in this paper is based on the Riemann–Liouville integral (see Appendix). A second order numerical solution of the fractional derivative of order q for values $0 < q < 1$ is given by Soon et al. [36], and it reads:

$$D^q \boldsymbol{x}_n = \frac{\Delta t^{1-q}}{\Gamma(3-q)} \sum_{p=0}^{n} a_{p,n} D^1 \boldsymbol{x}_p, \tag{3}$$

$$a_{p,n} = \begin{cases} (n-1)^{2-q} - n^{1-q}(n+q-2) \quad \text{if } p = 0, \\ \\ (n-p-1)^{2-q} - 2(n-p)^{2-q} + (n-p+1)^{2-q} \\ \text{if } 0 < p < n, \\ \\ 1 \quad \text{if } p = n, \end{cases}$$

where q is the derivative order such that $0 < q < 1$, n is the index of the most recently computed timestep, $a_{p,n}$ is the weight of the past timestep p computed at the current timestep n, Γ is the Gamma function, and $D^1 \boldsymbol{x}_p = \boldsymbol{v}_p$ is the velocity of the particle at past timestep p.

The fractional derivative of the position at the current time step is thus computed based on the velocities of the past, and the past velocities are prioritized by weights. The near past has more influence on the current state than the far past. The evolution of the weights generated by the fractional derivative operator is shown in Figure 2 for q values of 0.2, 0.5 and 0.8.

We compute the total spring force as the sum of forces generated by the history-based spring element and the regular elastic spring:

$$\boldsymbol{F} = -k_{min}\, \boldsymbol{x} - k_{hist} \left\| D^q \boldsymbol{x} \right\| \boldsymbol{x}. \tag{4}$$

Substituting Equation 3 into Equation 4 and rearranging, we obtain:

$$\boldsymbol{F} = -1 \left(k_{hist} \frac{\Delta t^{1-q}}{\Gamma(3-q)} \left\| \sum_{p=0}^{n} a_{p,n} \boldsymbol{v}_p \right\| + k_{min} \right) \boldsymbol{x}, \tag{5}$$

where k_{hist} is the control parameter for the history-based spring, $a_{p,n}$ is the weight of the past timestep p computed at the current timestep n, $\boldsymbol{v}_p$ is the velocity of the particle at the past timestep p, k_{min} is the stiffness constant of the elastic spring, and $\boldsymbol{x}$ is the vector of elongation or compression.

The above formulation successfully models both of the desired features of shear thickening fluids. First, the spring immediately becomes very stiff when the velocity $\boldsymbol{v}_p$ dramatically increases in the near past, what results in the mass-spring system exhibiting solid-like behavior in regions where great impact forces are applied. Second, the spring will gain memory thanks to the weights $a_{p,n}$ that apply to the velocities of the past. This results in the overall stiffness of the springs gradually rising under continuous application of forces, and gradually decreasing under the absence of forces. The

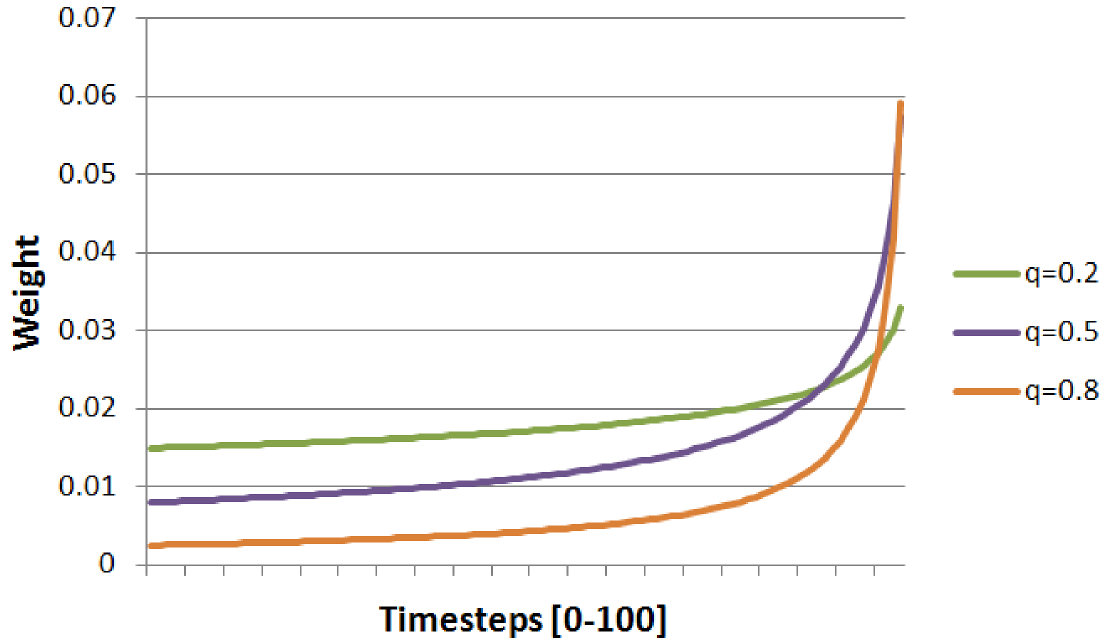

Figure 2: Weights produced by fractional derivatives of order 0.2, 0.5 and 0.8.

overall behavior achieved by the spring-mass system will lead to the shear thickening fluid slowly changing its phase both from liquid to solid and from solid to liquid.

Although k_{hist} and k_{min} are both denoted with the letter k in accordance with the Hooke's law, they serve different purposes. Constant k_{min} is the stiffness of the elastic spring and it specifies the minimum stiffness that the spring can have. The history-based stiffness originating from the fractional derivative is an additional term on top of the default stiffness. In the absence of great velocities, notice that the first term in the parenthesis of Equation 5 vanishes and the system behaves exactly like a regular viscoelastic fluid as described in [27]. However, when the particle gains instantaneous or cumulative high velocity over time, the first term adds extra stiffness to the spring. Constant k_{hist} is the parameter that controls the amount of extra stiffness that will be created due to the velocity increase. Constant k_{hist} can be used to give to the animator the ability to implicitly control the maximum stiffness that the system can gain. In cases where the integration method cannot keep the system numerically stable because of excess of stiffness, k_{hist} can be fine-tuned to keep the simulation stable.

In our several experiments the system stayed numerically stable when k_{hist} was kept up to 10 times the value of k_{min}. We used this as the maximum limit for a stable k_{hist}. For higher values of k_{hist}, the simulated material in our examples would eventually become too stiff. On the other hand if the value of k_{hist} was set too low, the force responses would not be sufficiently strong for the target shear-thickening phenomena to emerge. In our presented results we have used k_{hist} as 10 times the value of k_{min} in order to maximize the shear-thickening responses. The smaller the value the more the fluid looks like a standard viscoelastic fluid.

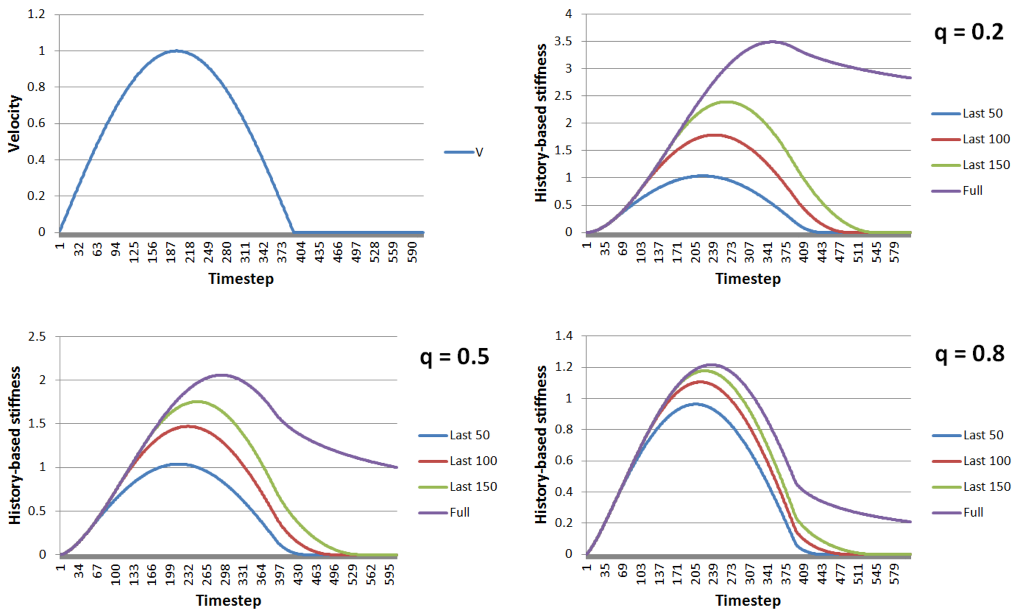

Figure 3: The upper-left graph shows the velocity of a single particle subjected to a temporary velocity increase due to an impact. The next three graphs (in left-right, top-down order) show the evolution of the history-based stiffness values of the spring connected to the same particle, when its velocity changes according to the upper-left graph. Curves for various d values are given in each graph. The second, third and forth graphs are based on the order of the fractional derivative $q = 0.2$, $q = 0.5$ and $q = 0.8$ respectively.

One important numerical drawback of the fractional derivative formulation is that it is a global derivative over the whole history of the particle. Clearly, this would pose performance problems as the simulation moves forward since the cost of calculating each simulation step would be always greater than the cost of the previous step. Therefore we limit the amount of history to be considered by changing the start index of the sum in Equation 5 from $p = 0$ to $p = n - d$, where d is the number of time steps in the past that we would like to take into account. Parameter d is also a useful parameter to model the characteristics of the material. If the material being modeled has long-term memory, it makes sense to use a large number of past states. On the other hand, if the material has short-term memory, a small value for d should be chosen.

We have investigated the behavior of a single particle under the influence of our history-based spring element in order to analyze the evolution of the history-based stiffness with respect to the particle's velocity magnitude. Figures 3 and 4 present several history-based stiffness values generated by various memory settings (various d values) and for history-based spring terms with various q values. Figure 3 demonstrates how the history-based stiffness changes when the particle gains a temporarily high velocity due to an impact. Figure 4 shows the evolution of the history-based stiffness when the particle moves with a steady velocity for a longer period time. It is possible to observe that the history-based stiffness increases more and lasts for a longer period when more states in the past are considered. Furthermore, it can be seen in the graph that limiting the number of past timesteps

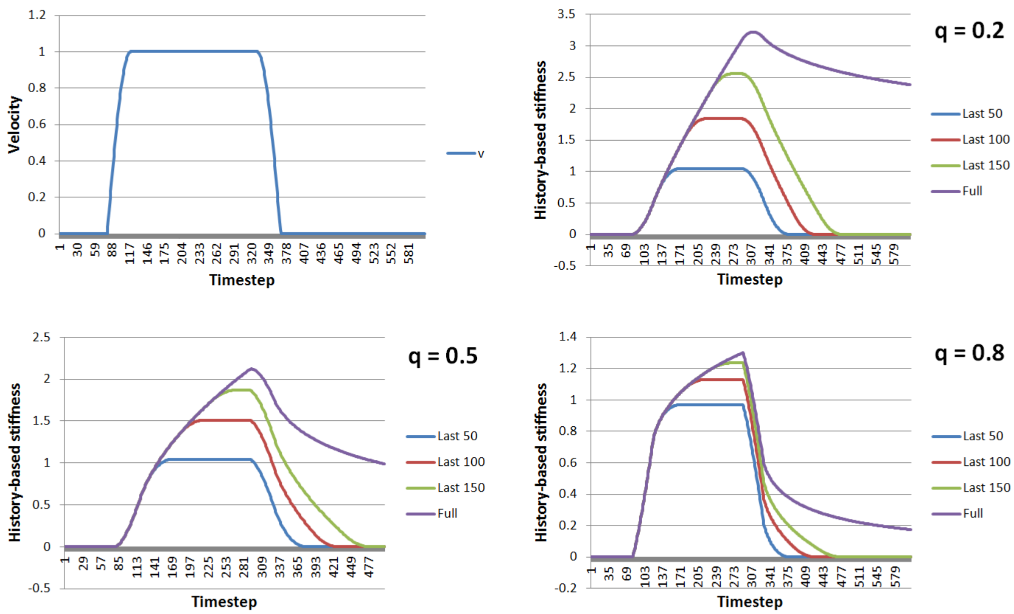

Figure 4: The shown graphs present the same plots as in Figure 3, except for the upper-left graph which shows the velocity of a single particle moving with constant velocity for a given period of time.

naturally creates a maximum limit for history-based stiffness, leading to obvious advantages in terms of controlling the numerical stability.

Another important modeling decision is the choice of the fractional derivative order. Equation 3 is defined for values of q such that $0 < q < 1$. The evolution of the weights generated by the fractional derivative for different values of q is presented in Figure 2. It can be observed that as q gets closer to 1, the history effects tend to disappear. As the value of q gets close to 0, the effect of the past time steps tend to become similar. Both extreme cases do not seem to be useful for our problem. Previous work on memory systems with fractional derivatives is based on using half-derivatives ($q = 0.5$) for describing a number of phenomena in rheology [37] [38] [39] [40] [36]. The value of $q = 0.5$ was also observed to produce best results in our experiments and we have used $q = 0.5$ as the fractional derivative order in our results.

Different values of q can be always specified by animators in order to experiment with different history effects. For the sake of examining the different behaviors that can be achieved, an analysis of the history effects generated with $q = 0.2$, $q = 0.5$, and $q = 0.8$ is also presented in Figures 3 and 4. We also present results demonstrating a longer-lasting solidification effect that can be achieved with $q = 0.2$ (Figure 8).

Considering all discussed parameters and adapting Equation 5 to the case of two particles connected by a spring, the total force between two given connected particles is:

$$\boldsymbol{F} = -1 \left(k_{hist} \frac{\Delta t^{0.5}}{\Gamma(2.5)} \left\| \sum_{p=n-d}^{n} a_{p,n} \left(\boldsymbol{v}_{pi} - \boldsymbol{v}_{pj} \right) \right\| + k_{min} \right) \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|} \left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\| - L \right), \tag{6}$$

where $\boldsymbol{v}_{pi}$ and $\boldsymbol{v}_{pj}$ are the velocities of the particles i and j at the past timestep p and L is the rest length of the spring. As part of the employed prediction-relaxation integration scheme [27], the displacement generated by the spring force is computed by multiplying the spring force by Δt^2, and then the displacement is applied to the positions of the particles.

3.3 Boundary Conditions

Experiments with real shear thickening fluids show that shear thickening can support objects with extreme masses. A bowling ball can jump and roll, and a person can literally walk on a shear thickening fluid while the fluid is in its solid-like state. Experiments in physics and rheology stress the importance of solid container walls in other shear thickening behavior [6]. The force applied to the surface is eventually transmitted to the boundaries of the container and solid walls help the fluid to support even greater masses.

Our network of history-based springs can simulate this behavior up to some extent. To improve the overall results an additional technique is employed for making sure that the particles at the boundary of the fluid practically stick to the boundary walls and mostly create reaction forces towards the interior instead of dissipating in multiple directions. For a particle at the boundary, its velocity is separated into its normal and tangential components. We completely cancel the normal velocity component that is perpendicular to the wall and going outwards. We also lower the tangential velocity component in a proportional way to the average of the history-based stiffness of the springs that the particle is connected to. In this way, when the fluid exhibits solid-like behavior, the particles at the boundaries are almost fixed. Later as the stiffness of the whole system decreases, the particles at the boundaries gradually return to their original behavior.

This technique allows our fluid to carry great masses during its solid-like phases. Our method ensures that particles do not escape the boundary and at the same time generate reaction forces towards the interior of the container. This scheme achieved the needed reaction at the boundary to support the solid-like phases of the fluid.

3.4 Summary of Overall Algorithm

The main steps of the overall simulation algorithm are summarized in Algorithm 1.

Time integration is computed through prediction-relaxation. In the prediction-relaxation approach, first the forces originating from gravity and viscosity are calculated, and then the velocity is updated based on these forces. The particle is then virtually moved to its predicted position based on the current velocity. At this predicted position, the density forces of SPH and the spring forces that account for the elasticity are calculated and the displacement due to these forces is immediately applied to the particle.

Lastly, collisions between the particles and the walls of the container are resolved and the positions of the particles are corrected. The velocities of the particles are implicitly calculated at the end of the timestep.

Algorithm 1 Simulation Step

// Apply gravity:
for all particle i **do**
    $\boldsymbol{v}_i \leftarrow \boldsymbol{v}_i + \Delta t \boldsymbol{g}$
end for
// Update velocities with viscosity forces:
ApplyViscosity();
for all particle i **do**
    $\boldsymbol{x}_i^{prev} \leftarrow \boldsymbol{x}_i$ // Save previous position
    $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_i + \Delta t \boldsymbol{v}_i$ // Advance to predicted position
end for
// Add or remove springs, or change rest lengths if necessary:
AdjustSprings();
// Apply displacements due to history-based spring elements:
ApplySpringDisplacements();
// Apply displacements due to pressure forces:
DoubleDensityRelaxation();
// Use previous positions to compute next velocities:
for all particle i **do**
    $\boldsymbol{v}_i \leftarrow (\boldsymbol{x}_i - \boldsymbol{x}_i^{prev})/\Delta t$
end for
// Take into account the boundary conditions:
DetectAndProcessBoundaryParticles();

4 Results and Discussion

We have evaluated our model with respect to achieving the interesting peculiar behavior of discontinuous shear thickening fluids and with respect to the how the results are changed according to the parameters of the model and in comparison to traditional viscoelastic fluids. Our simulations are available in the videos accompanying this paper.

4.1 Discontinuous Shear Thickening Behavior

There is an abundance of real experiments involving the most interesting phenomena observed in shear thickening fluids, namely with cornstarch and water mixtures, that show us how the material

Figure 5: Vibration experiment with a shear thickening fluid. The images show a comparison between the real experiment results [41] (left image) and our results obtained in simulation (right).

Figure 6: The snapshots from the vibration experiment with our shear thickening fluid model show the expected finger-like formations with various shapes. These formations do not appear if the history terms are removed.

behaves in different conditions. We have evaluated our model by visually comparing our simulations with these real experiments.

In our first experiment we have simulated a scene where a bowling ball is rolled on the surface of the shear thickening fluid. The real version of this experiment is important in many aspects. First, it shows that the fluid can support a high mass object such as a bowling ball and that the solid-like phase of the material can even make the bowling ball slightly bounce on the surface. Second, the bowling ball can roll on the fluid surface as if rolling on a solid surface. Third, even when the motion of the bowling ball ends, it still stays on the surface of the fluid for a given time, before it slowly starts to sink.

The bowling ball scene clearly shows the history effects in action in the mechanics of the fluid, and our model is able to successfully replicate in simulation the same behavior observed in real experiments. The fluid immediately solidifies under the high impact of the ball, acting somewhat like a spring mattress. It then keeps its solid-state for a while, letting the bowling ball roll on the surface, and when the ball stops the fluid gradually softens and enters into its liquid state allowing the ball to sink. These phases are illustrated by the snapshots presented in Figure 13. Without the history terms the ball just sinks, as shown in Figure 7.

In our second experiment we have simulated the shear thickening fluid subjected to vibration forces. In the real experiments the fluid is placed on a speaker creating vibrations that are transmitted to the fluid. After some time the fluid slowly starts changing its phase from liquid-like to solid-like.

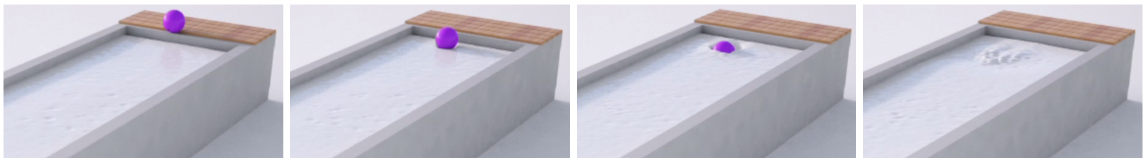

Figure 7: Without the history terms the bowling ball quickly sinks as in a normal fluid.

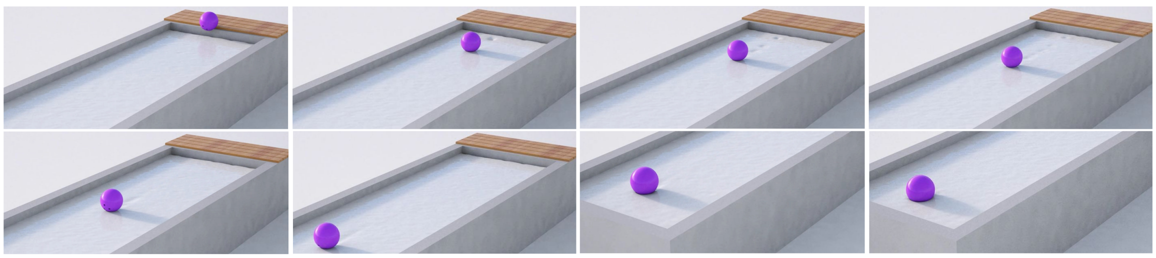

Figure 8: The snapshots show the long-lasting solidification effect achieved using a full history (i.e. using all the past timesteps) along with $q = 0.2$.

First the fluid forms holes and starts clustering at some random regions. Then the clustered regions slowly grow and form finger-like formations. The finger-like formations gradually get taller until they either merge with other parts of the fluid or break off from the main fluid body. Figure 5-left shows a snapshot of the real experiment.

We have performed a similar experiment in simulation with a fluid with 10K particles. We have placed equally distributed vibration sources at the bottom of the container and we have added random changes in the vibration frequency throughout the simulation. Figure 5 compares the behavior observed in the real experiment against the results obtained in simulation with our method. Additional results are presented in Figure 6, showing that our fluid model with history effects is able to successfully simulate the expected finger-like formations.

We have also compared the fluid with history-based springs against the fluid with no history terms. While, the vibration forces cause an accumulated history-based stiffness on the springs and the particles naturally start to climb on each other, the fluid without history terms could not achieve any of the finger-like formations. The comparison is presented in the accompanying video.

Our vibration experiment shows that finger-like formations require history terms, in agreement with the model of [7]. Our obtained results confirm that prediction in a dynamic simulation. The fact that some of the most interesting phenomena associated with shear thickening behavior can be simulated with our history-based model suggests that hysteresis may be more generally important to shear thickening than previously indicated. These effects have been ignored in the standard models for shear thickening such as hydroclustering [4] or dilatancy [6] and this insight may lead to improvements of those models so that they can finally predict some of the interesting phenomena associated with shear thickening.

4.2 Parameters and Comparisons

We have also analyzed the parameter space of our model. The main significant parameter is the fractional order q. In our model when $q = 0.5$ (half-derivative) the history effects are most visible. The other variations of the parameter q showed different in-between behaviors between solid and liquid, but they were not as visually interesting. Nevertheless, these variations are numerically demonstrated and the behavior obtained with different values of q is compared in Figures 3 and 4.

The second important parameter, d, is the number of timesteps used to compute the half derivative in Equation 6. This parameter plays an important role in controlling the duration of the transition from solid to liquid state. In order to better explore the role of the parameter d, we have simulated the fluid falling onto a flat surface and sliding out of an inclined surface with various d values. We have used a relatively large value of d ($d = 250$) for long history, a medium value ($d = 150$) for regular history and a relatively small value ($d = 50$) for short history effects. The obtained results are illustrated in Figures 9, 10, and 11.

Both the fluid drop and the fluid sliding experiments showed that larger values of d lead to a stiffer fluid after impact. As the value of d increases, the fluid takes relatively more time to go back to its liquid state and a long lasting solid state is achieved. Another example of a long lasting solid state can be seen in the last bowling ball animation (Figure 8). In comparison to the observed behavior of viscoelastic fluids, shear thickening fluids behave much like solids under high energy forces, leading to different effects which can be clearly observed in Figure 10, where the shear thickening behavior makes the fluid to clearly bounce up the surface in reaction to the first contact, something that is not at all observed without the history effects.

A larger number of possibilities that can be achieved by different combinations of d and q, are presented numerically in the graphs of Figures 3 and 4.

4.3 Performance

In terms of performance, the running time obtained with our system is approximately 1.5 seconds per frame for a simulation of 20K particles in an Intel Core(TM) i7-2600K 3.4 GHz computer. The added computation time by the inclusion of history-based spring forces is minimal and these terms do not affect the complexity of the algorithm. The weights of the fractional derivative term can be precomputed and used in combination with the stored past velocities. Equation 6 requires the algorithm to store the past velocities of the particles only up to a given number of timesteps, leading to an extra memory space requirement that is linear to the considered history size. For a history of d timesteps, d velocity vectors (and pre-computed scalar weights) are stored.

Table 1 summarizes computational times taken to compute the history terms within our system, showing that our proposed history-based effects can be obtained with a relatively small addition to the computation time taken by the underlying viscoelastic fluid solver.

In our implementation pressure forces are computed in the same way as described in Clavet's model [27]. The neighborhood search also follows Clavet's approach. It is based on a hashing grid which keeps track of neighborhood information per grid cell, and it is thus reduced to quickly accessing the infor-

Table 1: Computation time with respect to history size. Notation: d is the number of timesteps used in the computation of the half derivative, n is the number of particles, $t_{simstep}$ is the average computation time of one simulation step per frame without history terms, and $t_{history}$ is the average time per frame to compute the history terms. Time $t_{simstep}$+$t_{history}$ includes all computations necessary to advance the system by one timestep, which was fixed at 0.033 seconds. The last column shows the amount of extra memory needed to store the past velocities of the history terms, and was computed as $m_{history} = 12nd$, where 12 is the number of bytes used to store one velocity vector.

Experiment	d	n	$t_{simstep}$	$t_{history}$	$m_{history}$
Bowling Ball	0	20K	1.226 s	0 ms	0
Bowling Ball	50	20K	1.335 s	4 ms	12MB
Bowling Ball	150	20K	1.494 s	12 ms	36MB
Bowling Ball	500	20K	1.627 s	40 ms	120MB
Vibration	0	20K	1.192 s	0 ms	0
Vibration	50	20K	1.245 s	4 ms	12MB
Vibration	150	20K	1.388 s	12 ms	36MB
Vibration	500	20K	1.539 s	40 ms	120MB

mation in the cell containing a given particle, as described by Teschner *et al.* [42]. The computation time for neighborhood search is negligible.

4.4 Discussion

While only a given number of past timesteps are used in the computation of the fractional derivative term, the obtained value is an approximation of the true fractional derivative value and as such it can be as close as needed to the real value. The computation results in a weighted combination of past velocities which resembles a local smoothing or convolution method defined by the fractional derivative formulation (Equation 3). Most importantly this formulation enables the most interesting behavior of shear-thickening fluids to emerge in simulation (Figures 1 and 5). When our model is applied to simulations that do not include high energy interactions, the model produces a fluid that resembles a standard viscoelastic fluid. See Figure 12 for an example.

Our model has shown to successfully simulate the interesting effects of shear-thickening fluids, which can be observed in Figure 1 with the fluid supporting a heavy object during its high-energy phase, and in Figure 5 where the induced solid behavior enables the fluid to build structures on itself. For instance, our bowling ball scenario achieves the suspension in solid-like behavior for a relatively long time (with a yield stress exceeding the balls weight per unit area) until the bowling ball slows down enough that the suspension gradually becomes liquid again, during some sort of a transition period between liquid and solid-like behavior. For instance, a modified Herschel-Bulkley model would result in a more gradual sinking of the ball, and would miss the achieved oscillations in the velocity of the sinking object.

Our method builds on a literature-grounded model to achieve these unique effects and is also relatively simple to implement given that it is based on a particle-based model. Additional techniques like considering surface tension could be experimented for improving some of the results, but we also choose to keep our model as simple as possible and all presented results are directly generated from the presented equations. We believe the achieved effects open new directions for creating deformable materials that transition between fluid and solid behavior, a notion that can be explored by designers in order to create new types of fluid effects or animated fluid-based characters.

In comparison to models for viscoelastic fluids, our simulations show that our model clearly transitions from solid-like behavior under high energy forces to a fluid behavior as energy dissipates. This difference between a typical viscoelastic behavior and our model is for example depicted in Figure 10, where the shear thickening behavior makes the fluid to mostly preserve its shape while clearly bouncing up a surface in reaction to falling on it, and then to gradually transition to a regular viscoelastic behavior. While some viscoelastic materials (like for example silly putty) may also bounce like in the examples shown in Figures 10 and 11, a simulation model would need to incorporate a history-dependent mechanism on the elastic term of the model in order to allow a transition to a more fluid-like behavior. This would achieve a behavior where the elastic stress decays over a relaxation time after which the viscous behavior takes over and the material becomes more fluid. Currently simulation models for viscoelastic materials do not include such a behavior. In addition, the effects illustrated in Figures 1 and 5 also cannot be achieved with normal viscoelastic fluids.

4.5 Limitations and Future Work

As for limitations, although our model is able to imitate shear thickening fluids to a great extent, we have observed that in the vibration experiment the finger-like formations did not grow as tall as in the real experiment. This prevented the replication of finger-like formations falling and merging with each other, and forming bridges.

Although we have not addressed shear thinning fluids and yield stress fluids in the scope of this paper, there is a wide literature on using fractional calculus for the simulation of these kinds of fluids as well. Yield stress fluids typically have hysteresis in the yield stress, although the hysteresis is in the opposite direction than for shear thickening fluids (higher viscosity after a history of resting). We believe these fluids can be simulated with variations of our model.

Another direction for future work is to extend our approach to formulations based on Position-Based Dynamics (PBD) [33] [34], which represent an increasingly popular approach for efficient fluid simulation. One first direction to approach such an extension is to add fractional terms to the density constraint solver. For instance, the density constraint is usually written as $C(\boldsymbol{p} + \Delta\boldsymbol{p}) = 0$ [34], which is enforced by a series of Newton steps along the constraint gradient. Instead of simply using the gradient, the gradient vector could be multiplied by fractional derivative terms in order to achieve a history-based multiplier that could be tuned to achieve shear thickening behavior. Another possible area to be explored is to add fractional terms to the XSPH viscosity term that is also incorporated [34].

5 Conclusions

We have introduced a new approach that is able to visually simulate some interesting peculiar behavior of shear thickening fluids. The proposed model is based on an efficient history-based stiffness term which has showed to be essential for achieving discontinuous shear thickening behavior.

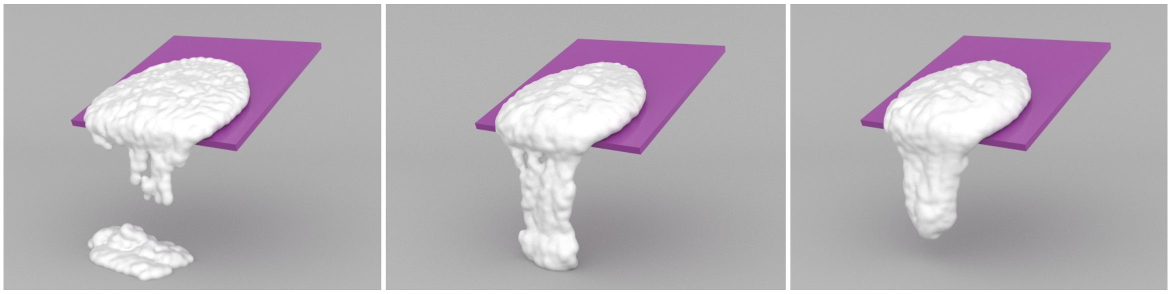

Figure 9: These snapshots are taken at the same timestep and show different solidification durations of fluids using various history size (d = 50, d = 150, d = 250).

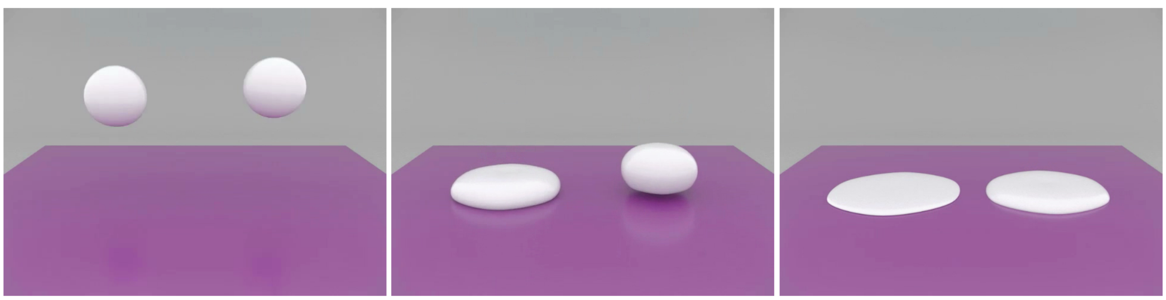

Figure 10: These snapshots show a comparison between the different behavior obtained without history terms, thus equivalent to a viscoelastic fluid (left ball), and with our history terms modeling a shear-thickening fluid (right ball). The fluid on the left hits the surface and gradually looses its shape, without major reactions, until reaching a stable form as expected for a viscoelastic fluid tuned to display that behavior. In contrast, the fluid on the right is a shear-thickening fluid and, due to the relatively high-energy impact, it first behaves much as a solid bouncing up from the surface one time before it starts to loose energy and transition to the regular viscoelastic behavior.

The obtained results were able to reproduce particularly interesting examples that can be observed in real experiments. Our results show that complex relationships between hysteresis, viscoelasticity and flows can be successfully reproduced with a relatively simple and efficient SPH-based model. More generally we demonstrate new techniques for simulating unconventional fluid behavior in a visually plausible way.

6 Appendix

The possibility of defining non-integer derivatives has been raised since 1695, when l'Hôpital sent a letter to Leibniz asking for a definition for a half derivative. Today, Fractional Calculus is the branch of mathematical analysis that studies differentiation of non-integer order. Different definitions for a non-integer derivative have been developed [43]. The numerical formulation adopted in this paper is based on the popular Riemann–Liouville integral:

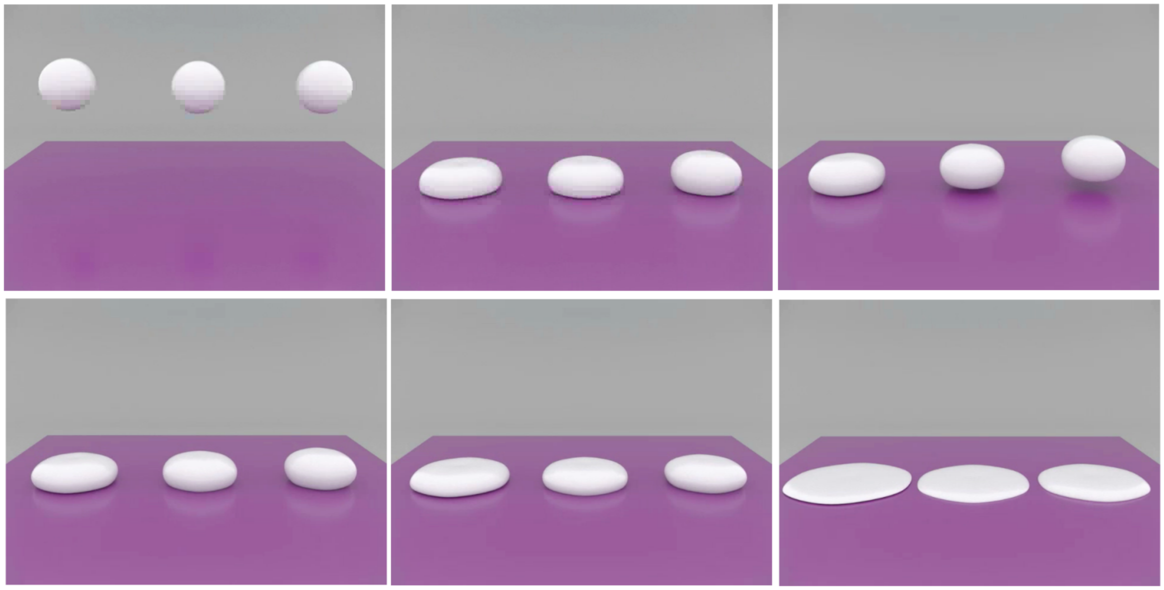

Figure 11: The snapshots are taken from the comparison simulation of fluids using various history size ($d = 50$, $d = 150$, $d = 250$). The fluid on the right has the highest history size, therefore it slightly jumps after the first surface contact and keeps its solid state for the longest time.

Figure 12: In simulations that do not include high energy interactions the result resembles a standard viscoelastic fluid.

$$D^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} f(\sigma) d\sigma, \tag{7}$$

where α is the derivative order, and Γ is the generalized factorial function.

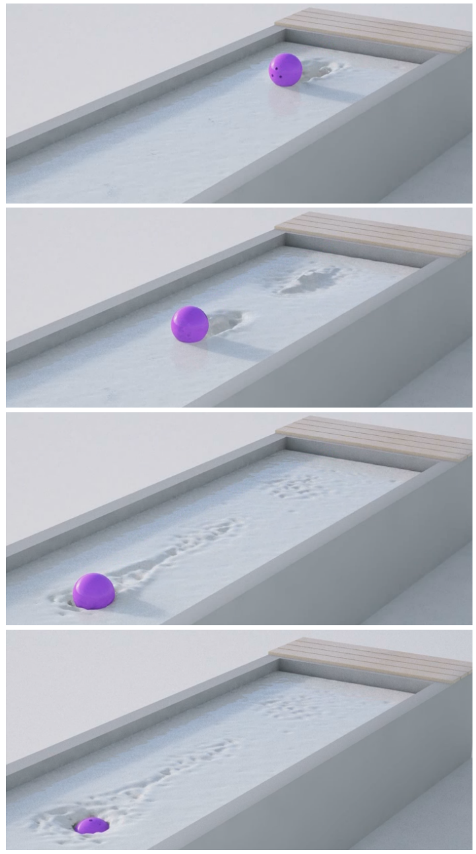

Figure 13: The first image (in top-down order) shows the first moment of impact where the shear thickening fluid successfully carries the mass of the bowling ball and makes it bounce. The second image shows the second moment of impact after the bounce. The bowling ball still won't sink but slows down. The third image shows the moment where the bowling ball comes to a stop. Since the fluid is still in its solid-like state, the bowling ball will not immediately sink. The fourth image shows the end of the simulation where the history-based stiffness starts to vanish and the fluid transitions to fluid-like state, finally resulting in the sinking of the bowling ball.

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